The connection of Monge-Bateman equations with ordinary differential equations and their generalisation

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Abstract

It is shown that the Monge equation is equivalent to the ordinary differential equation $\ddot{X}=0$ of free motion. Equations of Monge type (with their general solutions) are connected with each ordinary differential equation of second order $\ddot{X}=F(\dot{X},X;t)$, integrable by quadratures. The result is generalised to a system of equations of the second order, which is in one to one correspondence with the multidimensional Monge-Bateman system.

1 Introduction

The famous Monge equation, which has been quoted in textbooks for more than 150 years, has the form:

$$\lambda_t = \lambda\lambda_x$$

Its implicit solution:

$$x - \lambda t = f(\lambda)$$

reminds one of the solution of the equation of free motion with constant velocity λ with initial value for the coordinate $x_0 = f(\lambda)$ at $t = 0$.

This similarity is not accidental and in the present note we want to draw attention to the fact of the connection of the Monge equation and the generalised it Bateman equation [1]:

$$(\frac{\lambda_t}{\lambda_x})_t - (\frac{\lambda_t}{\lambda_x})(\frac{\lambda_t}{\lambda_x})_x = 0 \tag{1}$$

with the ordinary differential equation $\ddot{X} = 0$. This fact allow us to introduce partial differential equations connected with each ordinary differential equation of second order (this limitation is inessential) which possess the same integrable properties as the initial ODE.

This result can be generalised to any system of equations of second order, leading to a generalisation of the Monge and Bateman type of equations to the multidimensional case. The hydrodynamic system of D.B.Fairlie [2] is the simplest example of such a generalisation.

2 The main assertion and its proof

Assertion

Let $\ddot{X} = F(\dot{X}, X; t)$ be an ordinary differential equation, where F is an arbitrary function of its arguments and $\dot{}$ denotes differentiation with respect to the independent argument t. Then the equation in partial derivatives:

$$(\frac{\lambda_t}{\lambda_X})_t - (\frac{\lambda_t}{\lambda_X})(\frac{\lambda_t}{\lambda_X})_X = F(-\frac{\lambda_t}{\lambda_X}, X; t) \tag{2}$$

is exactly integrable (implicitly) simultaneously with the initial ordinary differential equation.

We would like to prove this assertion from both sides. First by using the known solution of the initial ordinary differential equation and secondly by explicit exchange of variables in it directly.

2.1 The first proof

The general solution of an ordinary differential equation of second order depends upon two arbitrary constants (c^1, c^2) which we will consider as functions of two arguments $\lambda(X,t)$. So we have an implicit definition of the function λ:

$$X = X(t; c^1(\lambda), c^2(\lambda)) \tag{3}$$

By $'$ we denote the derivative of X with respect to the argument λ:

$$X' = X_{c^1} c^1_\lambda + X_{c^2} c^2_\lambda$$

It is obvious that dot and prime differentiations are mutually commutative. We obtain in consequence after differentiation of (3) with respect to its independent arguments X and t:

$$1 = X'\lambda_X \equiv (X_{c_1} c_1' + X_{c_2} c_2')\lambda_X, \quad 0 = X_t + X'\lambda_t$$

or

$$X_t = -\frac{\lambda_t}{\lambda_X}$$

Differentiation of the last equation once more by the arguments X, t leads to the result:

$$X_{tt} + X_t'\lambda_t = -(\frac{\lambda_t}{\lambda_X})_t, \quad X_t'\lambda_X = -(\frac{\lambda_t}{\lambda_X})_X \tag{4}$$

Eliminating X_t' from the last two equalities and taking into account that under differentiation X_{tt} c^1, c^2 remain constant we arrive at the generalised Bateman equation (2). This way from a given solution to equation it satisfied is decribed in each manual.

The generalized Monge equation may be obtain from (2) after making the identification $\frac{\lambda_t}{\lambda_X} \to \lambda$ and takes the form:

$$\lambda_t - \lambda\lambda_X = F(-\lambda, X; t) \tag{5}$$

2.2 The second proof

Let us present the solution of equation of the second order from the assertion in the form $Q(X,t) = \text{constant}$. Then for derivatives $\dot{X}$ (with the help of the theorem of differentiation of implicit functions) we obtain:

$$\dot{X} = -\frac{Q_t}{Q_X}$$

Repeating this operation once more we come to the generalized Bateman equation from this assertion (with the obvious exchange $Q \to \lambda$).

The corollary of the results of last two subsections may be summarised in:

Proposition (equivalent to the main assertion):

Each equation of second order can be presented in Monge-Bateman form (2). If the general solution of an ODE may be presented in explicit form, then the general solution of (5) is given by (3).

3 Multidimensional generalisation

Suppose we are given a set of $(n-1)$ arbitrary functions $X^i \equiv X^i(c^\alpha;t)$, each one depending on $2(n-1)$ variables c^α and a single "time" variable t. All $2(n-1)$ independent variables c^α may be expressed implicitly in terms of a system of $2(n-1)$ equations:

$$x^i = X^i(c^\alpha;t), \quad \dot{x}^i = \frac{\partial X^i(c^\alpha;t)}{\partial t} \tag{6}$$

After substitution of these values into explicit expressions for second derivatives we arrive at a system of ordinary differental equations of second order (as described in textbooks):

$$x^i_{tt} = \frac{\partial^2 X^i}{\partial t^2}(x, x_t; t)$$

Now let us consider $2(n-1)$ values c^α as a functions of $(n-1)$ functions $\rho^s \equiv \rho^s(x;t)$. Differentiating the first equation of (6) first with respect to x_j and secondly with respect to t we obtain respectively:

$$\delta_{ij} = \sum X^i_{\rho^s} \rho^s_{x_j}, \quad 0 = \sum X^i_{\rho^s} \rho^s_t + X^i_t$$

From the last equations we obtain immediately:

$$\{X^i_{\rho^s}\} = J^{-1}(\rho; x), \quad X_t = -J^{-1}\rho_t$$

where $J(\rho; x)$ is the Jacobian matrix. Further differentiation of the second equation with respect to x_i, t arguments leads to the result:

$$\sum X^i_{\rho^s,t}\rho^s_{x_i} = -(J^{-1}\rho_t)_{x_i}, \quad X^i_{tt} + \sum X^i_{\rho^s,t}\rho^s_t = -(J^{-1}\rho_t)t$$

Eliminating the matrix $\{X^i_{\rho^s,t}\}$ from the last system we obtain finally:

$$\tau^s_t - \sum \tau^r \tau^s_{x_r} = X^s_{tt}(x, \tau; t) \tag{7}$$

where $\tau = J^{-1}\rho_t$.

In the case of a noninteracting system $X^s_{tt} = 0$ (7) goes over to the so called hydrodynamical system introduced and solved by D.B.Fairlie [2]. In connection with this, the above implicit solution of the hydrodynamical system arises after treating the constants of motion in the trajectories of n free moving particles as functions of n arbitrary functions ρ. Thus n equations:

$$x^s = f^s(\rho)t + g^s(\rho)$$

define in implicit form the solution of the n dimensional Bateman-Monge equations. If we choose $f^s = \rho^s$ we reproduce the original form of the solution of the hydrodynamical system proposed in [2].

4 Outlook

The results contained in formulae (4),(5) and (7) are so simple and clear that they don't demand additional comments. It is possibile that they have been discovered long ago, but the author has not seen them in the literature.

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